

# An Extended Table of Roots of $J_n'(\alpha) Y_n'(\beta\alpha) - J_n'(\beta\alpha) Y_n'(\alpha) = 0$

By John F. Bridge and Stanley W. Angrist

An eigen-equation that frequently occurs in mathematical physics involving an annular cavity is

$$(1) \quad J_n'(x) Y_n'(\beta x) - J_n'(\beta x) Y_n'(x) = 0$$

where  $J_n(x)$  and  $Y_n(x)$  are respectively Bessel functions of the first and second kinds.

J. McMahon [1] gave an asymptotic expression for the roots to this equation, D. O. North [2] obtained a root smaller than the first root given by the asymptotic expression of McMahon, and R. Truell [2] developed a graphical method for obtaining this root.

H. B. Dwight [3] gave the first six roots of equation (1) for values of  $n$  from 1 to 3 and for various values of  $\beta$  from 1 to 4.

The purpose of this paper is to extend the range and accuracy of the roots to equation (1) and to determine the ranges of the solutions for which the asymptotic expression proposed by McMahon is sufficiently accurate.

The calculation of the roots was accomplished by trial and error substitution in the following equation:

$$(2) \quad F_n(x) = J_n'(x) Y_n'(\beta x) - J_n'(\beta x) Y_n'(x).$$

Starting with  $x = 0.1$ ,  $F_n(x)$  was calculated, increasing  $x$  in steps of 0.1 until  $F_n(x)$  changed sign. A linear interpolation was then used to determine the approximate value of the root, then the Newton-Raphson iteration procedure was used until two successive approximations of the root value were within  $\pm 10^{-6}$ . The root thus obtained was compared with the root obtained with all four terms of J. McMahon's asymptotic expression (3)

$$(3) \quad \begin{aligned} x_n^{(s)} &= \delta + \frac{p}{\delta} + \frac{q - p^2}{\delta^2} + \frac{r - 4pq + 2p^3}{\delta^5} + \dots \\ \delta &= \frac{(s-1)}{\beta-1}, \quad p = \frac{m+3}{8\beta}, \quad m = 4n^2 \\ q &= \frac{4(m^2 + 46m - 63)(\beta^3 - 1)}{3(8\beta)^3(\beta - 1)} \\ r &= \frac{32(m^3 + 185m^2 - 2053m + 1899)(\beta^5 - 1)}{5(8\beta)(\beta - 1)}. \end{aligned}$$

If the two values were within  $\pm 2 \times 10^{-5}$ , the asymptotic value was used for that root and all larger roots for the given value of  $n$ . If not, the procedure was continued until the next root was found and compared with its asymptotic value.

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Received January 26, 1961. The research reported in this article was sponsored in part by the Rome Air Development Center, Air Research and Development Command.

TABLE 1  
 $\beta = 1.1$

$s \setminus n$	1	2	3	4	5	6	7	8	9	10	11
1	0.952738	31.44125A	62.84451A	94.25622A	125.67004A	157.08470A	188.49978A	219.91511A	251.33059A	282.74616A	314.16181A
2	1.905471	31.48469A	62.86622A	94.27069A	125.68089A	157.09338A	188.50702A	219.92131A	251.33602A	282.75098A	314.16615A
3	2.858190	31.55696A	62.90238A	94.29480A	125.69898A	157.10785A	188.51907A	219.93165A	251.34506A	282.75902A	314.17338A
4	3.810883	31.65787A	62.95298A	94.32855A	125.72429A	157.12810A	188.53595A	219.94611A	251.35772A	282.77027A	314.18851A
5	4.763541	31.78713A	63.01796A	94.37192A	125.75683A	157.15414A	188.55765A	219.96471A	251.37399A	282.78474A	314.19653A
6	5.716165	31.94442A	63.09730A	94.42490A	125.79659A	157.18596A	188.58417A	219.98745A	251.39388A	282.80242A	314.21245A
7	6.668739	32.12983A	63.19094A	94.48747A	125.84356A	157.22555A	188.61550A	220.01430A	251.41739A	282.82332A	314.23125A
8	7.621261	32.34139A	63.29881A	94.55962A	125.89774A	157.26691A	188.65165A	220.04529A	251.44451A	282.84743A	314.25295A
9	8.573720	32.580055	63.42085A	94.64133A	125.96911A	157.31604A	188.69261A	220.08041A	251.47524A	282.87474A	314.27754A
10	9.526102	32.844769	63.56071A	94.73257A	126.02767A	157.37094A	188.73838A	220.11965A	251.50958A	282.90528A	314.30502A
11	10.478414	33.134922	63.70707A	94.83330A	126.10339A	157.43159A	188.78895A	220.16301A	251.54753A	282.93902A	314.33539A
12	11.430636	33.449843	63.87107A	94.94351A	126.18629A	157.49799A	188.84432A	220.21050A	251.58909A	282.97596A	314.36865A

TABLE 2  
 $\beta = 1.2$

$s \setminus n$	1	2	3	4	5	6	7	8	9	10	11
1	0.910334	15.75443A	31.43914A	47.13936A	62.84346A	78.54910A	94.25552A	109.96238A	125.66951A	141.37683A	157.08428A
2	1.820564	15.83425A	31.47819A	47.16590A	62.86336A	78.56502A	94.26878A	109.97374A	125.67946A	141.38567A	157.09223A
3	2.730596	15.96642A	31.54522A	47.21009A	62.89651A	78.59154A	94.29088A	109.99269A	125.69604A	141.40041A	157.10550A
4	3.640327	16.14971A	31.63775A	47.27189A	62.94289A	78.62866A	94.32182A	110.01921A	125.71924A	141.42104A	157.12406A
5	4.549656	16.382436	31.75632A	47.35123A	63.00247A	78.67635A	94.36158A	110.05330A	125.74907A	141.44755A	157.14793A
6	5.458492	16.662640	31.90066A	47.44803A	63.07522A	78.73461A	94.41015A	110.09494A	125.78552A	141.47996A	157.17710A
7	6.366728	16.988042	32.07043A	47.56217A	63.16109A	78.80340A	94.46752A	110.14414A	125.82858A	141.51824A	157.21156A
8	7.274269	17.356215	32.26522A	47.69354A	63.26002A	78.88270A	94.53368A	110.20088A	125.87825A	141.56240A	157.25131A
9	8.181018	17.764617	32.484585	47.84200A	63.37197A	78.97248A	94.60860A	110.26515A	125.93452A	141.61244A	157.29636A
10	9.086883	18.210665	32.728057	48.00738A	63.49685A	79.07271A	94.69226A	110.33694A	125.99731A	141.66834A	157.34668A
11	9.991769	18.691814	32.995112	48.18953A	63.63460A	79.18333A	94.78465A	110.41623A	126.06637A	141.73010A	157.40228A
12	10.895584	19.205560	33.285178	48.38825A	63.78513A	79.30432A	94.88373A	110.50301A	126.14282A	141.79771A	157.46317A

TABLE 3  
 $\beta = 1.5$

$s \setminus n$	1	2	3	4	5	6	7	8	9	10	11
1	0.805089	6.376507	12.61286A	18.88052A	25.15596A	31.43450A	37.71458A	43.99556A	50.27708A	56.55898A	62.84113A
2	1.608064	6.538074	12.69279A	18.93368A	25.19579A	31.46635A	37.74112A	44.01830A	50.29698A	56.57667A	62.85705A
3	2.406845	6.800078	12.82502A	19.02197A	25.26205A	31.51937A	37.78532A	44.05619A	50.33013A	56.60614A	62.88857A
4	3.199502	7.153187	13.008069	19.14495A	25.35455A	31.59347A	37.84711A	44.10917A	50.37651A	56.64737A	62.92069A
5	3.984302	7.586740	13.240072	19.30201A	25.47301A	31.68849A	37.92641A	44.17721A	50.43608A	56.70034A	62.96837A
6	4.759868	8.089989	13.518772	19.492366	25.61710A	31.80427A	38.02313A	44.26023A	50.50879A	56.76502A	63.02861A
7	5.525283	8.652747	13.841692	19.715153	25.78642A	31.94060A	38.13713A	44.35816A	50.59460A	56.84136A	63.09537A
8	6.280183	9.265874	14.206206	19.969378	25.980511	32.09722A	38.26828A	44.47091A	50.69344A	56.92934A	63.17463A
9	7.024764	9.920594	14.609666	20.253958	26.198870	32.27387A	38.41641A	44.59836A	50.80524A	57.02889A	63.26433A
10	7.759723	10.610011	15.049482	20.567770	26.440936	32.47023A	38.58134A	44.74040A	50.92992A	57.13996A	63.36445A
11	8.486088	11.326711	15.523169	20.909628	26.706139	32.685987	38.76286A	44.89691A	51.06740A	57.26248A	63.47494A
12	9.205064	12.065020	16.028379	21.278336	26.993792	32.920781	38.96077A	45.06774A	51.21756A	57.396639A	63.59574A

TABLE 4  
 $\beta = 2.0$

$s \setminus n$	1	2	3	4	5	6	7	8	9	10	11
1	0.677337	3.282470	6.35320A	9.47132A	12.60124A	15.73584A	18.87278A	22.01105A	25.15015A	28.28981A	31.42985A
2	1.340602	3.531292	6.474705	9.55158A	12.66121A	15.78374A	18.91265A	22.04521A	25.18003A	28.31636A	31.45374A
3	1.978877	3.920058	6.673802	9.684216	12.76068A	15.86331A	18.97896A	22.10205A	25.22976A	28.36057A	31.49853A
4	2.587614	4.418954	6.946140	9.867677	12.89893A	15.97417A	19.07148A	22.18142A	25.29925A	28.42236A	31.54916A
5	3.169444	4.992926	7.286813	10.100009	13.075002	16.11553A	19.18991A	22.28313A	25.38837A	28.50165A	31.62057A
6	3.731081	5.613492	7.690964	10.379013	13.287800	16.28765A	19.33386A	22.40693A	25.49695A	28.59832A	31.70767A
7	4.279317	6.254733	8.153854	10.702460	13.536067	16.488875	19.50289A	22.55254A	25.62478A	28.71221A	31.81035A
8	4.819109	6.898225	8.670590	11.068269	13.818503	16.718760	19.696490	22.71963A	25.77164A	28.84318A	31.92849A
9	5.353551	7.532315	9.233539	11.474624	14.133790	16.976439	19.914120	22.907796	25.93728A	28.99102A	32.06196A
10	5.884431	8.151579	9.834129	11.920106	14.480688	17.261049	20.156198	23.116669	26.12141A	29.15554A	32.21059A
11	6.412784	8.755433	10.460403	12.405537	14.858080	17.571711	20.419105	23.345816	26.32737A	29.33652A	32.37423A
12	6.939212	9.346029	11.100036	12.923681	15.265067	17.907606	20.705234	23.594788	26.54393A	29.53371A	32.55269A

TABLE 5  
 $\beta = 2.5$

$\begin{matrix} s \\ n \end{matrix}$	1	2	3	4	5	6	7	8	9	10	11
1	0.584712	2.263639	4.273300	6.33922A	8.41951A	10.50548A	12.59427A	14.68467A	16.77607A	18.86813A	20.96067A
2	1.136960	2.566404	4.422384	6.436661	8.49197A	10.56321A	12.64227A	14.72575A	16.81198A	18.90004A	20.98937A
3	1.643266	3.014111	4.666427	6.597438	8.6111963	10.65901A	12.72201A	14.79406A	16.87173A	18.95314A	21.03715A
4	2.112840	3.541025	5.000385	6.819653	8.778449	10.792257	12.8313A	14.88935A	16.9515A	19.02731A	21.10392A
5	2.561246	4.090006	5.417247	7.101488	8.990216	10.962206	12.97514A	15.01131A	17.06200A	19.12239A	21.18956A
6	2.998819	4.626484	5.903285	7.441703	9.246103	11.168015	13.14745A	15.15952A	17.19202A	19.23818A	21.28391A
7	3.430580	5.139039	6.434476	7.839507	9.545276	11.40883A	13.34943A	15.33555A	17.34488A	19.37444A	21.41681A
8	3.858792	5.630601	6.981766	8.292655	9.887485	11.684016	13.580621	15.532939	17.52022A	19.53090A	21.55804A
9	4.284519	6.107923	7.522039	8.793645	10.273141	11.993085	13.840098	15.757181	17.717707	19.70728A	21.71739A
10	4.708335	6.576321	8.044625	9.327038	10.702781	12.336095	14.127730	16.005829	17.936898	19.903310	21.89462A
11	5.130594	7.038968	8.549352	9.872603	11.175293	12.713618	14.443191	16.278474	18.177451	20.11856A	22.08947A
12	5.551540	7.497588	9.040527	10.412958	11.684992	13.126721	14.786590	16.574836	18.439022	20.35285A	22.30169A

TABLE 6  
 $\beta = 3.0$

$\begin{matrix} s \\ n \end{matrix}$	1	2	3	4	5	6	7	8	9	10	11
1	0.513621	1.757764	3.236112	4.774941	6.32990A	7.89128A	9.45582A	11.02216A	12.58962A	14.15782A	15.72655A
2	0.977493	2.090108	3.406676	4.885065	6.411255	7.95586A	9.50941A	11.06797A	12.62963A	14.19385A	15.75850A
3	1.388031	2.542188	3.687198	5.067624	6.546166	8.063062	9.59844A	11.14413A	12.69619A	14.25247A	15.81167A
4	1.769223	3.019714	4.068117	5.322325	6.734053	8.212376	9.722532	11.25038A	12.78910A	14.33503A	15.88596A
5	2.137715	3.473757	4.520694	5.649721	6.974869	8.403356	9.88128A	11.38637A	12.90812A	14.44085A	15.98123A
6	2.500224	3.899285	4.998066	6.047230	7.269718	8.635923	10.074332	11.551802	13.052973	14.56971A	16.09731A
7	2.859234	4.306567	5.462847	6.499287	7.620584	8.910762	10.301571	11.746359	13.223370	14.721406	16.234020
8	3.215799	4.703812	5.903976	6.974669	8.026997	9.229644	10.563347	11.969928	13.419092	14.895667	16.391141
9	3.570477	5.095160	6.326594	7.443281	8.478551	9.595008	10.860782	12.222663	13.640002	15.092317	16.568487
10	3.923627	5.482493	6.737917	7.892390	8.951907	10.007154	11.195926	12.505215	13.886161	15.311226	16.76553A
11	4.275300	5.866719	7.142516	8.323863	9.421931	10.458332	11.571339	12.818914	14.157920	15.552398	16.98251A
12	4.626286	6.248353	7.542723	8.743594	9.875703	10.929875	11.987573	13.165952	14.456107	15.816020	17.21912A

TABLE 7  
 $\beta = 3.5$

$\frac{s}{n}$	1	2	3	4	5	6	7	8	9	10	11
1	0.457116	1.454468	2.615215	3.837282	5.07677A	6.32324A	7.57314A	8.82498A	10.07803A	11.33188A	12.58630A
2	0.851944	1.790848	2.803828	3.957933	5.165169	6.393081	7.63093A	8.87430A	10.12106A	11.37006A	12.62061A
3	1.195733	2.218507	3.112756	4.160117	5.312582	6.509334	7.727080	8.95637A	10.19268A	11.43361A	12.67773A
4	1.518391	2.624121	3.510574	4.446164	5.520210	6.672168	7.861501	9.07104A	10.29275A	11.52241A	12.75757A
5	1.832864	2.990998	3.933706	4.811544	5.791144	6.882622	8.034366	9.218229	10.421139	11.63633A	12.86001A
6	2.143190	3.350464	4.336324	5.226722	6.128080	7.143320	8.246529	9.398148	10.577829	11.775335	12.984989
7	2.450808	3.694364	4.713594	5.645665	6.523097	7.458223	8.500044	9.611471	10.762977	11.939352	13.132395
8	2.756407	4.032861	5.075430	6.043274	7.024514	7.827645	8.798504	9.859776	10.977151	12.128544	13.302238
9	3.060412	4.367606	5.429045	6.420705	7.355345	8.237044	9.145286	10.145968	11.221559	12.343591	13.494622
10	3.363111	4.698379	5.777776	6.785951	7.748037	8.656638	9.536486	10.473877	11.498474	12.584795	13.710019
11	3.664716	5.028646	6.123040	7.144385	8.124571	9.062966	9.932988	10.844846	11.811287	12.864542	13.949078
12	3.965386	5.355740	6.465500	7.498594	8.491429	9.451361	10.368570	11.250050	12.163235	13.155423	14.213202

TABLE 8  
 $\beta = 4.0$

$\frac{s}{n}$	1	2	3	4	5	6	7	8	9	10	11
1	0.411126	1.251118	2.202078	3.212742	4.241767	5.27817A	6.31824A	7.36038A	8.40380A	9.44806A	10.49292A
2	0.752324	1.589316	2.405800	3.342808	4.336187	5.352370	6.37944A	7.41249A	8.44920A	9.48830A	10.52905A
3	1.048387	1.966515	2.730106	3.563697	4.495331	5.476593	6.481572	7.499362	8.52486A	9.55534A	10.58925A
4	1.329098	2.310028	3.109408	3.875323	4.723462	5.652598	6.625298	7.621222	8.63082A	9.64917A	10.67349A
5	1.603863	2.627489	3.473731	4.247062	5.024741	5.884420	6.812303	7.778725	8.767350	9.769892	10.781799
6	1.875312	2.932111	3.810141	4.621730	5.385800	6.177375	7.046238	7.973473	8.935118	9.917781	10.914280
7	2.144462	3.232994	4.130526	4.971975	5.764014	6.527116	7.332444	8.208688	9.135594	10.093496	11.071230
8	2.411858	3.528861	4.443063	5.303097	6.124802	6.904143	7.671331	8.489368	9.371626	10.298392	11.253282
9	2.677860	3.821681	4.751048	5.624179	6.465090	7.272555	8.043947	8.818328	9.647736	10.534963	11.461681
10	2.942723	4.111963	5.055736	5.939872	6.793059	7.621048	8.417143	9.184515	10.007262	10.807262	11.698622
11	3.206628	4.400066	5.357709	6.252056	7.114497	7.955208	8.772984	9.559798	10.326486	11.119591	11.967695
12	3.469713	4.686272	5.657324	6.561496	7.432047	8.281445	9.113000	9.921876	10.701430	11.470218	12.273217

TABLE 9  
 $\beta = 4.5$

$\frac{s}{n}$	1	2	3	4	5	6	7	8	9	10	11
1	0.373031	1.104180	1.907419	2.767026	3.645583	4.531899	5.42204A	6.31436A	7.20801A	8.10256A	8.99772A
2	0.672262	1.429128	2.122859	2.905834	3.745536	4.609929	5.486135	6.36881A	7.25537A	8.14448A	9.03533A
3	0.932730	1.761353	2.447073	3.143448	3.916412	4.741800	5.593741	6.459896	7.334463	8.21444A	9.09807A
4	1.181574	2.058531	2.788328	3.465085	4.164959	4.931864	5.746831	6.588549	7.44574A	8.31263A	9.18605A
5	1.425681	2.366965	3.099201	3.808736	4.484038	5.186972	5.949805	6.756880	7.590729	8.439631	9.299559
6	1.666949	2.607632	3.390988	4.128994	4.827895	5.503764	6.209304	6.969205	7.770095	8.596503	9.439164
7	1.906187	2.873843	3.672789	4.428087	5.153729	5.846841	6.524074	7.231840	7.989542	8.785455	9.606028
8	2.143874	3.136779	3.948702	4.716739	5.458655	6.176491	6.865922	7.544832	8.254520	9.010520	9.802340
9	2.380321	3.397053	4.223228	5.000127	5.752218	6.485955	7.198239	7.885246	8.565928	9.277316	10.031957
10	2.615753	3.655079	4.494004	5.280120	6.040102	6.783352	7.511375	8.219436	8.904830	9.587293	10.300197
11	2.850334	3.911170	4.762411	5.557443	6.324555	7.074637	7.811930	8.535612	9.240321	9.924668	10.608892
12	3.084192	4.165575	5.028733	5.832455	6.606423	7.362401	8.105941	8.838861	9.559061	10.261035	10.944743

TABLE 10  
 $\beta = 5.0$

$\frac{s}{n}$	1	2	3	4	5	6	7	8	9	10	11
1	0.341023	0.992169	1.686615	2.433009	3.198648	3.972324	4.74998A	5.52992A	6.31124A	7.09349A	7.87637A
2	0.606945	1.298988	1.909895	2.579945	3.303850	4.053914	4.816710	5.586423	6.36030A	7.13684A	7.91523A
3	0.839813	1.592758	2.221209	2.829861	3.486480	4.193557	4.929637	5.681463	6.442497	7.209368	7.98018A
4	1.063471	1.854676	2.522036	3.142537	3.750957	4.398621	5.092836	5.817096	6.55894A	7.31164A	8.07151A
5	1.283122	2.103700	2.794283	3.445672	4.064092	4.672712	5.313819	5.997813	6.712042	7.445044	8.190073
6	1.500255	2.346950	3.053043	3.723205	4.367847	4.985908	5.594879	6.230930	6.906519	7.612121	8.337250
7	1.715570	2.586472	3.305774	3.987439	4.648429	5.289542	5.907917	6.517340	7.149364	7.817850	8.515920
8	1.929486	2.823103	3.554787	4.245629	4.916186	5.571978	6.211067	6.830058	7.440007	8.068763	8.731140
9	2.142291	3.0527348	3.800915	4.500246	5.177982	5.842169	6.494648	7.132534	7.752285	8.362823	8.988886
10	2.354177	3.289371	4.044605	4.752137	5.436350	6.106535	6.766599	7.416815	8.053985	8.674564	9.285744
11	2.573055	3.520053	4.286170	5.001701	5.692160	6.367605	7.032887	7.690071	8.338672	8.975433	9.596870
12	2.775771	3.749018	4.525860	5.249210	5.945795	6.626274	7.296003	7.957848	8.612915	9.260326	9.896883

The roots calculated by the McMahan expression (3) are indicated in the tables by the letter A printed after the number. It should be noted that equation (3) does not give a root for  $s = 1$ .

The great speed of the IBM 704 digital computer was used to advantage in the solution of this problem. The Bessel functions were generated by using recursion relationships. Starting with an arbitrarily small number for a Bessel function of high order for a given argument, successively smaller orders were calculated until Bessel functions of 6- to 7-place accuracy were obtained. It was not possible with the particular procedure used to evaluate functions with arguments larger than 50.

The authors wish to express their appreciation for the advice given by Professor L. S. Han who originally suggested the problem.

Department of Mechanical Engineering  
Ohio State University  
Columbus, Ohio

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